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A first-quantized formalism for cosmological particle production

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Abstract: Given suitable boundary conditions, we show that the initial state and the amount of particle production in a cosmological spacetime are encoded in the Feynman propagator. The propagator can be represented in terms of a particle path integral in an auxiliary spacetime, and particle production can be extracted from the auxiliary propagator. This provides a first-quantized formalism for computing cosmological particle production which, unlike conventional Bogolubov transformations, may be amenable to a string-theoretic generalization.

Keywords: Bosonic Strings, Space-Time Symmetries, Cosmology of Theories beyond the SM.

Contents

1. Introduction

The standard approach to cosmological particle production [1, 2] involves finding two sets of solutions to the wave equation, with pair production given by a Bogolubov transformation between the two sets of modes. Although this method has a long and proven record of success, there are reasons to search for alternative techniques. In particular, the second-quantized formalism underlying Bogolubov transformations does not carry over to string theory, whose standard perturbative formulation is first-quantized. In order to estimate, say, the spectrum of density fluctuations coming from stringy effects [3]–[8], one would like to have a first-quantized technique in which one could systematically compute α' corrections.

In this paper we develop a first-quantized formalism for computing pair-production in an arbitrary cosmological background. The method is based on the Feynman propagator. It turns out to be convenient to work in terms of the propagator in an auxiliary 'reflected' spacetime. We show that pair production is encoded in this auxiliary propagator. Moreover the auxiliary propagator admits a straightforward path integral representation, so our formalism allows us to compute particle production without using Bogolubov transformations. Note that first-quantized methods have been used to study pair creation from horizons [9, 10], where again Bogolubov transformations are usually used. First-quantized techniques for particle production have also been developed in [11, 12].

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An outline of this paper is as follows. In section 2 we give a brief review of the standard treatment of a scalar field in an FRW universe. In section 3 we consider the Feynman propagator. We discuss the way in which initial conditions and particle production are encoded in the propagator. We then give a path integral representation of the auxiliary propagator. Next, we present some compact formulas for the total amount of particle production. Finally, we show that the propagator in the original spacetime can be constructed from the auxiliary propagator using the method of images. In section 4 we apply our techniques to the case of a cosmology with a power-law scale factor. We conclude in section 5.

2. Scalar field quantization

Consider a free minimally-coupled real scalar field $\hat{\phi}(x)$ in a Robertson-Walker universe with metric

$$ds^2 = a^2(\eta)\left(-d\eta^2 + h_{ij}dx^i dx^j\right). \tag{2.1}$$

Here $a(\eta)$ is the scale factor, η is conformal time, and h_{ij} is the metric of a maximally-symmetric n-dimensional space. We make the field redefinition

$$\hat{\phi}(x) = a^{(1-n)/2}(\eta)\hat{\chi}(x)\,, \tag{2.2}$$

and write $\hat{\chi}(x)$ as a sum over momentum modes:

$$\hat{\chi}(x) = \sum_k \left(\hat{a}_k f_k(\vec{x})u_k(\eta) + \hat{a}_k^\dagger f_k^*(\vec{x})u_k^*(\eta)\right). \tag{2.3}$$

The corresponding mode functions $u_k(\eta)$ satisfy

$$u_k'' + V_k u_k = 0 \tag{2.4}$$

where

$$V_k(\eta) = k^2 + a^2m^2 - \frac{1}{2}(n-1)\frac{a''}{a} - \frac{1}{4}(n-1)(n-3)\left(\frac{a'}{a}\right)^2. \tag{2.5}$$

Here $'$ indicates a derivative with respect to η. If one thinks of η as position, then this is a one-dimensional Schrödinger equation with potential $-\frac{1}{2}V_k(\eta)$.

2.1 Final state

Throughout this paper we assume that the scale factor changes adiabatically in the future. Equivalently, we assume that the WKB approximation becomes valid as $\eta \to +\infty$, so we can take the modes to satisfy

$$u_k(\eta) \equiv u_k^{\text{out}}(\eta) \sim e^{-i\int^\eta d\eta' \sqrt{V(\eta')}} \qquad \text{as } \eta \to +\infty\,. \tag{2.6}$$

That is, we take $u_k(\eta)$ to be positive-frequency in the future. Thus the operators $\hat{a}_k$ annihilate the preferred future, or out, vacuum:

$$\hat{a}_k|\text{out}\rangle = 0 \qquad \forall k\,. \tag{2.7}$$

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2.2 Initial state

We also need to specify the initial state of the field. First choose a conformal time η_0 at which to fix initial conditions. Then for each momentum k choose a complex parameter ω_k. Where no physical principle dictates the choice of initial state, the ω_k are just arbitrary parameters, aside from the fact that (for reasons given below) $\mathrm{Re}\,\omega_k > 0$ and $\omega_k = \omega_{-k}$. Now let $\hat{\chi}_k(\eta)$ be a Fourier component of the field,

$$\hat{\chi}_k(\eta) = \int dx\, f_k^*(\vec{x})\hat{\chi}(\eta,\vec{x}) = \hat{a}_k u_k(\eta) + \hat{a}_{-k}^\dagger u_{-k}^*(\eta)\ . \tag{2.8}$$

To specify the initial state of the field we define the operators

$$\hat{b}_k \equiv \sqrt{\frac{\omega_k\,|\omega_k|}{2\,\mathrm{Re}\,\omega_k}}\left(\hat{\chi}_k(\eta_0) + \frac{i\hat{\pi}_k(\eta_0)}{\omega_k}\right)\ . \tag{2.9}$$

Here the conjugate momentum $\hat{\pi}_k(\eta) = \partial_\eta\hat{\chi}_k$ satisfies $i[\hat{\pi}_k,\hat{\chi}_{k'}] = \delta_{k+k'}$ as well as $\hat{\chi}_k^\dagger = \hat{\chi}_{-k}$, $\hat{\pi}_k^\dagger = \hat{\pi}_{-k}$. The normalization of $\hat{b}_k$ and the condition $\omega_k = \omega_{-k}$ ensure that $[\hat{b}_k,\hat{b}_{k'}^\dagger] = \delta_{kk'}$, $[\hat{b}_k,\hat{b}_{k'}] = [\hat{b}_k^\dagger,\hat{b}_{k'}^\dagger] = 0$. We take the initial state to satisfy $\hat{b}_k|\mathrm{in}\rangle = 0$, or equivalently

$$\partial_\eta\hat{\chi}_k(\eta_0)|\mathrm{in}\rangle = i\omega_k\hat{\chi}_k(\eta_0)|\mathrm{in}\rangle\ . \tag{2.10}$$

For example, in Minkowski space, the standard Poincaré-invariant vacuum satisfies this relation with $\omega_k = \sqrt{k^2+m^2}$. Other choices are possible: for example the Rindler vacuum with acceleration s corresponds to setting $\omega_k = k\coth(\pi k/2s)$.

We do not assume that the universe is adiabatic at early times. Thus (unlike the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$) there may be no useful sense in which the operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ actually annihilate and create particles. Rather we just use these operators to specify the initial state of the field. An equivalent way to characterize the initial state is to note that, as a consequence of (2.10), the initial wavefunctional for the field has a general gaussian form, of the sort familiar in Schrödinger picture field theory [13]:

$$\Psi[\chi]|_{\eta=\eta_0} \sim \exp\left(-\frac{1}{2}\sum_k \omega_k|\chi_k|^2\right)\ . \tag{2.11}$$

The widths of the gaussian are determined by the adjustable parameters ω_k; we must have $\mathrm{Re}\,\omega_k > 0$ so that the wavefunctional is normalizable.

The space of initial states we can consider in this manner is quite large: the values of the parameters ω_k are essentially arbitrary, so we can specify one complex parameter per momentum mode. Thus we can sweep out the entire space of states with wavefunctions of the form (2.11).[1] Equivalently we can sweep out the entire space of states that are related by Bogolubov transformations.[2] Moreover, our formalism can easily be extended to non-vacuum initial states of the form $(\hat{b}_k^\dagger)^n|\mathrm{in}\rangle$, by following the analysis in [14].

[1] Such states were referred to as 'Fock vacua' in [13].

[2] Although two complex parameters α_k and β_k appear in the Bogolubov transformations (2.12), they are constrained to satisfy $|\alpha_k|^2 - |\beta_k|^2 = 1$. Also a Bogolubov transformation which merely multiplies $\hat{b}_k$ by a phase is redundant since it leaves $|\mathrm{in}\rangle$ invariant. Hence the physical content of a Bogolubov transformation is captured by one complex parameter.

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2.3 Bogolubov transformations

We now recall some standard results [1, 2]. Bogolubov transformations express the operators $\hat{a}_k$, which annihilate the out-vacuum, in terms of $\hat{b}_k$, and vice versa:

$$\hat{a}_k = \sum_{k'} \left(\alpha_{k'k} \hat{b}_{k'} + \beta^*_{k'k} \hat{b}^\dagger_{k'} \right) , \qquad \hat{b}_k = \sum_{k'} \left(\alpha^*_{kk'} \hat{a}_{k'} - \beta^*_{kk'} \hat{a}^\dagger_{k'} \right) . \tag{2.12}$$

Equivalently, they express the positive-frequency out-modes in terms of the positive- and negative-frequency in-modes, and vice versa:[3]

$$u^{\rm out}_k = \sum_{k'} \left(\alpha^*_{k'k} u^{\rm in}_{k'} - \beta_{k'k} u^{\rm in\,*}_{k'} \right) , \qquad u^{\rm in}_k = \sum_{k'} \left(\alpha_{kk'} u^{\rm out}_{k'} + \beta_{kk'} u^{\rm out\,*}_{k'} \right) . \tag{2.13}$$

The spatial modes are orthonormal, so the Bogolubov coefficients are diagonal in k:

$$\alpha_{kk'} \equiv \alpha_k \delta_{kk'} \qquad \beta_{kk'} \equiv \beta_k \delta_{k,-k'} . \tag{2.14}$$

The initial state can be written as a squeezed state [15, 16],

$$|{\rm in}\rangle = \prod_k C_k \exp\left(-\frac{1}{2} \gamma_k \hat{a}^\dagger_k \hat{a}^\dagger_{-k} \right) |{\rm out}\rangle , \tag{2.15}$$

where $C_k = (1 - |\gamma_k|^2)^{1/4}$ so that $\langle {\rm in}|{\rm in}\rangle = 1$. Requiring that $|{\rm in}\rangle$ be annihilated by the $\hat{b}_k$ implies (using (2.12) and the commutation algebra) that $\gamma_k = -\beta^*_k/\alpha^*_k$. Note that, from (2.15),

$$\gamma_k = -\frac{\langle {\rm out}|\hat{a}_{-k}\hat{a}_k|{\rm in}\rangle}{\langle {\rm out}|{\rm in}\rangle} . \tag{2.16}$$

We see that γ_k is related to pair creation; indeed, it is the normalized probability amplitude for the initial state to evolve at late times into a state that has two particles in it, $\hat{a}^\dagger_k \hat{a}^\dagger_{-k}|{\rm out}\rangle$. The average number of particles produced is given by the expectation value of the number operator,

$$\langle {\rm in}|\hat{N}_k|{\rm in}\rangle = |\beta_k|^2 = \frac{|\gamma_k|^2}{1 - |\gamma_k|^2} , \tag{2.17}$$

where we have used the normalization condition $|\alpha_k|^2 - |\beta_k|^2 = 1$.

3. Particle production from the propagator

In this section we present our main results. We begin by discussing the way in which initial conditions and particle production are encoded in the Feynman propagator. Then we give a path integral representation of the Feynman propagator of an auxiliary spacetime. Along with the parameter ω that encodes the choice of initial state, this auxiliary propagator neatly yields the particle production for the given cosmology, as we then show. Finally, we use the method of images to give a first-quantized representation of the actual propagator in that cosmology.

[3] By positive-frequency in-mode we just mean the coefficient of $\hat{b}_k$ in the mode expansion of the field.

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3.1 The Feynman propagator

The Feynman propagator is defined as the vacuum expectation value of the time-ordered product of field operators. However, the vacua appearing on the left and right of the operator product can in general be different:

$$iG_F(k,\eta_1;k',\eta_2) = \frac{\langle \mathrm{out}|T\hat{\chi}_k(\eta_1)\hat{\chi}_{-k'}(\eta_2)|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} \,. \tag{3.1}$$

The amount of particle production gets encoded in the propagator. To see this, we write $\hat{\chi}_k$ in terms of modes that are positive-frequency at late times, $\hat{\chi}_k = \hat{a}_k u_k(\eta) + \hat{a}^\dagger_{-k} u^*_{-k}(\eta)$, and find that (for $\eta_1 > \eta_2$)

$$\begin{aligned} iG_F(k,\eta_1;k',\eta_2) &= \frac{\langle \mathrm{out}|\hat{a}_k\hat{a}^\dagger_{k'}|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} u_k(\eta_1)u^*_{k'}(\eta_2) + \frac{\langle \mathrm{out}|\hat{a}_k\hat{a}_{-k'}|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} u_k(\eta_1)u_{-k'}(\eta_2) \\ &= \left(u_k(\eta_1)u^*_k(\eta_2) - \gamma_k u_k(\eta_1)u_{-k}(\eta_2)\right)\delta_{kk'} \,. \end{aligned} \tag{3.2}$$

To obtain the second term in the second line we used (2.16). Since we can take $u_k(\eta) = u_{-k}(\eta)$ we will suppress all the k indices henceforth. Then in general the Feynman propagator, expressed in terms of the out-modes, is simply

$$iG_F(\eta_1,\eta_2) = u(\eta_1)u^*(\eta_2)\theta(\eta_{12}) + u(\eta_2)u^*(\eta_1)\theta(\eta_{21}) - \gamma u(\eta_1)u(\eta_2) \,. \tag{3.3}$$

The first two terms are standard but the third term, being proportional to γ, is a signal of pair creation; see (2.17).

The choice of initial state is also encoded in early-time boundary conditions on the propagator. To see this, we set the boundary conditions on iG_F at some time in the far past, η_0, using (2.10):

$$\partial_{\eta_2} iG_F(\eta_1,\eta_2)|_{\eta_2=\eta_0} = \frac{\langle \mathrm{out}|\hat{\chi}(\eta_1)(\partial_\eta\hat{\chi})(\eta_0)|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} = \frac{\langle \mathrm{out}|\hat{\chi}(\eta_1)(i\omega)\hat{\chi}(\eta_0)|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} \,. \tag{3.4}$$

Hence the choice of initial state fixes the boundary condition

$$\partial_{\eta_2} iG_F(\eta_1,\eta_2)|_{\eta_2=\eta_0} = (i\omega) iG_F(\eta_1,\eta_0) \qquad \text{for } \eta_1 > \eta_0 \,. \tag{3.5}$$

3.2 Auxiliary path integral

We have seen that the amount of particle production is encoded in the Feynman propagator; this makes it natural to seek a path integral representation for the propagator.

It is tempting to set

$$iG_F(\eta_1,\eta_2) \stackrel{?}{=} \langle \eta_1|\frac{i}{-D^2+i\epsilon}|\eta_2\rangle = \int_0^\infty ds\, e^{-\epsilon s} \int_{\eta(0)=\eta_2,\eta(s)=\eta_1} \mathcal{D}\eta(\cdot)\, e^{iS[\eta]} \,. \tag{3.6}$$

Here $D^2(\eta) = \partial^2_\eta + V(\eta)$ is the differential (Schrödinger) operator that acts on the modes in (2.4). We have introduced an $i\epsilon$ prescription and given a path integral representation involving a Schwinger proper-time parameter s and a worldline action

$$S[\eta] = \int_0^s dt \left(-\frac{1}{4}(\partial_t\eta)^2 - V(\eta)\right) . \tag{3.7}$$

There are two potential difficulties with this expression for the propagator. The first is that the potential $V(\eta)$ may be singular at early times, which could make the path integral ill-defined. The second is that, even if one can represent a Green's function in this way, there's no reason to expect it to satisfy the boundary conditions (3.5).

Rather than discuss iG_F directly, we avoid these difficulties by introducing an auxiliary spacetime. The propagator in the auxiliary spacetime $i\bar{G}_F$ admits a straightforward path integral representation, and as we show in the next section, particle production can be computed from $i\bar{G}_F$. We will work out a path integral representation for iG_F itself in section 3.4.

Thus consider a separate, auxiliary problem in which we reflect the spacetime symmetrically about η_0. We then have a new operator, $\bar{D}^2(\eta)$, defined by

$$\bar{D}^2(\eta) \equiv \begin{cases} D^2(\eta) & \eta \geq \eta_0 \\ D^2(2\eta_0 - \eta) & \eta \leq \eta_0 \, . \end{cases} \tag{3.8}$$

The corresponding potential $\bar{V}(\eta)$ is the reflection of V about η_0. We define the Green's function in this reflected problem with an $i\epsilon$ prescription, so that it can be computed in first-quantized terms.

$$\begin{aligned} i\bar{G}_F(\eta_1,\eta_2) &= \langle \eta_1 | \frac{i}{-\bar{D}^2 + i\epsilon} | \eta_2 \rangle = \int_0^\infty ds\, e^{-\epsilon s} \int_{\eta(0)=\eta_2,\eta(s)=\eta_1} \mathcal{D}\eta(\cdot)\, e^{i\bar{S}[\eta]} \\ \bar{S}[\eta] &\equiv \int_0^s dt \left(-\frac{1}{4} (\partial_t \eta)^2 - \bar{V}(\eta) \right) . \end{aligned} \tag{3.9}$$

Notice that, as we are throwing away the part of the potential before η_0, any early-time singularity of the potential is irrelevant. The reflected potential may have a cusp at η_0 but this causes no problems. A further advantage of reflecting the potential is that the reflected spacetime is asymptotically adiabatic in the past as well as in the future, so the $i\epsilon$ prescription automatically selects the preferred in- and out-vacua of the reflected problem. That is, as $\eta_1, \eta_2 \to \pm\infty$ the reflected Green's function obeys the adiabatic boundary conditions considered in section 2.1. For example[4]

$$\partial_{\eta_2} \log i\bar{G}_F(\eta_1,\eta_2) \simeq i\sqrt{\bar{V}(\eta_2)} \qquad \text{as } \eta_2 \to -\infty \, . \tag{3.11}$$

3.3 Formulas for particle production

We now derive some compact formulas for the amount of particle production. We will find two expressions, one that still refers to mode solutions and one which is expressed entirely in terms of the reflected propagator introduced above.

[4]The proof is as follows. The Feynman propagator $i\bar{G}_F(\eta_1,\eta_2) = \langle \eta_1 | \frac{i}{-\bar{D}^2 + i\epsilon} | \eta_2 \rangle$. Insert a complete set of energy eigenstates $\bar{D}^2|\omega\rangle = E(\omega)|\omega\rangle$ and let $\psi_\omega(\eta) = \langle \eta | \omega \rangle$. Adiabaticity implies that for $\eta \approx \eta_2$ we can take $\psi_\omega(\eta) \approx \exp(-i\omega\eta)$ with $E(\omega) \approx -\omega^2 + \bar{V}(\eta_2)$. Then

$$i\bar{G}_F(\eta_1,\eta_2) \approx \int_{-\infty}^{\infty} d\omega \rho(\omega) \frac{i}{\omega^2 - \bar{V}(\eta_2) + i\epsilon} \psi_\omega(\eta_1) e^{i\omega\eta_2} \tag{3.10}$$

where $\rho(\omega)$ is the density of states. The usual contour deformation picks up the pole at $\omega = \sqrt{\bar{V}(\eta_2)} - i\epsilon$, and (3.11) follows. Our approximations become exact as $\eta_2 \to -\infty$.

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What we would like is an expression for $|\gamma|^2$; see (2.17). As we have seen, this quantity appears in the propagator. Inserting the expression for the propagator (3.3) into the boundary conditions (3.5), we find that

$$u(\eta_1)u'^*(\eta_0) - \gamma u(\eta_1)u'(\eta_0) = i\omega\left(u(\eta_1)u^*(\eta_0) - \gamma u(\eta_1)u(\eta_0)\right) \tag{3.12}$$

which implies

$$\gamma = \frac{\omega + i\partial \ln u^*(\eta_0)}{\omega + i\partial \ln u(\eta_0)} \cdot \frac{u^*(\eta_0)}{u(\eta_0)} \,. \tag{3.13}$$

We can simplify this expression by defining

$$\bar{\omega} \equiv i\partial_\eta \ln u(\eta)\big|_{\eta=\eta_0} \,. \tag{3.14}$$

The amount of pair creation is determined by $|\gamma|^2$, which is given by[5]

$$|\gamma|^2 = \left|\frac{\omega - \bar{\omega}^*}{\omega + \bar{\omega}}\right|^2 . \tag{3.15}$$

This is the equation we are after. It tells us that particle production is completely determined by ω and $\bar{\omega}$. The initial state only enters through the parameter ω, so it's very easy to see how particle production depends on the choice of initial state. The final state, and the dynamics of the scale factor, only enter in determining $\bar{\omega}$. Thus $\bar{\omega}$ is the only thing one actually has to compute. It can be obtained from (3.14) without using the in-modes, by differentiating the out-modes at the point η_0 where initial conditions are specified.

Next we show that $\bar{\omega}$ can also be determined simply by taking a derivative of the reflected propagator, without knowing either the in-modes or the out-modes:

$$i\bar{\omega} = \partial_{\eta_2} \ln i\bar{G}_F(\eta_1,\eta_2)\big|_{\eta_1>\eta_2=\eta_0} \,. \tag{3.16}$$

Extracting $\bar{\omega}$ from the reflected propagator in this way and using (3.15) thus gives us a way of obtaining particle production without any reference to modes or Bogolubov transformations. To establish this, first note that the reflected Green's function can be expressed as

$$i\bar{G}_F(\eta_1,\eta_2) = \bar{u}(\eta_1)\bar{u}^*(\eta_2)\theta(\eta_{12}) + \bar{u}(\eta_2)\bar{u}^*(\eta_1)\theta(\eta_{21}) - \bar{\gamma}\bar{u}(\eta_1)\bar{u}(\eta_2) \,. \tag{3.17}$$

This is the analog of (3.3) for the reflected problem, where $\bar{u}(\eta)$ is a positive-frequency out-mode in the reflected problem, and where $\bar{\gamma}$ indicates particle production in the reflected spacetime. For $\eta \geq \eta_0$ we can set $\bar{u}(\eta) = u(\eta)$ so that

$$\partial_{\eta_2} \ln i\bar{G}_F(\eta_1,\eta_2)\big|_{\eta_1>\eta_2=\eta_0} = \partial_\eta \ln\left(u^*(\eta) - \bar{\gamma}u(\eta)\right)\big|_{\eta=\eta_0} \,. \tag{3.18}$$

Next note that a positive-frequency in-mode in the reflected problem is given by

$$\bar{u}^{\rm in}(\eta) = \begin{cases} u^*(2\eta_0 - \eta) & \eta \leq \eta_0 \\ \bar{\alpha}u(\eta) + \bar{\beta}u^*(\eta) & \eta \geq \eta_0 \end{cases} \tag{3.19}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are Bogolubov coefficients in the reflected problem (see (2.13)). The in-mode

[5] Although we based our derivation on the propagator, this result could have been obtained by other means.

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and its derivative are continuous at η_0, so we must have

$$\partial_\eta \ln u^*(2\eta_0 - \eta)|_{\eta=\eta_0} = \partial_\eta \ln(\bar{\alpha}u(\eta) + \bar{\beta}u^*(\eta))|_{\eta=\eta_0} . \tag{3.20}$$

Taking the complex conjugate of this equation, and recalling that $\bar{\gamma} = -\bar{\beta}^*/\bar{\alpha}^*$, we find that (3.18) can be written as

$$\partial_{\eta_2} \ln i\bar{G}_F(\eta_1,\eta_2)\big|_{\eta_1>\eta_2=\eta_0} = -\partial_\eta \ln u(\eta)|_{\eta=\eta_0} \equiv i\bar{\omega} \tag{3.21}$$

where we have used (3.14).

3.4 Method of images

At this point we have achieved our goal of finding a first-quantized formalism for calculating the amount of particle production. Nevertheless, it is amusing and instructive to express the original Green's function in terms of a sum over particle paths.

Recall that the original propagator was subject to nontrivial boundary conditions. As a first step, let us try to implement these boundary conditions using the method of images. We therefore write the original Green's function in terms of the reflected Green's function and an image charge q:

$$iG_F(\eta_1,\eta_2) = i\bar{G}_F(\eta_1,\eta_2) + q\, i\bar{G}_F(\eta_1, 2\eta_0 - \eta_2) . \tag{3.22}$$

For $\eta_1, \eta_2 > \eta_0$ note that iG_F is indeed a Green's function for the operator $D^2(\eta)$. It remains to implement the boundary condition at η_0 by choosing q appropriately. From (3.5) we require

$$\partial_{\eta_2} iG_F(\eta_1,\eta_2)|_{\eta_1>\eta_2=\eta_0} = (i\omega)iG_F(\eta_1,\eta_0) \tag{3.23}$$

while from (3.22) we have

$$\begin{aligned} iG_F(\eta_1,\eta_0) &= (1+q)i\bar{G}_F(\eta_1,\eta_0) , \\ \partial_{\eta_2} i\, G_F(\eta_1,\eta_2)|_{\eta_1>\eta_2=\eta_0} &= (1-q)\partial_{\eta_2} i\, \bar{G}_F(\eta_1,\eta_2)\big|_{\eta_1>\eta_2=\eta_0} . \end{aligned} \tag{3.24}$$

Thus the image charge is given by

$$q = \frac{\bar{\omega} - \omega}{\bar{\omega} + \omega} , \tag{3.25}$$

where we have invoked the definition of $\bar{\omega}$, (3.16).

The method of images leads to a way of representing the original Feynman propagator in terms of a particle path integral. Use (3.22) to represent iG_F as a sum of two particle path integrals in the auxiliary reflected potential. In each path integral fold the particle paths across $\eta = \eta_0$, to obtain particle worldlines that are restricted to satisfy $\eta(s) \geq \eta_0$. Note that this folding leaves the worldline action invariant. By adding the two path integrals one can represent iG_F as a single path integral, just as in (3.6), but where the particle paths are restricted to satisfy $\eta(s) \geq \eta_0$, and where the boundary conditions at $\eta = \eta_0$ are enforced by weighting paths according to the rule

$$\begin{array}{ll} e^{iS[\eta]} & \text{for paths that touch } \eta_0 \text{ an even number of times} \\ qe^{iS[\eta]} & \text{for paths that touch } \eta_0 \text{ an odd number of times} . \end{array} \tag{3.26}$$

Similar constructions to enforce boundary conditions on the path integral have been discussed in [17, 18].

This approach leads to a nice conceptual visualization. The formation of a pair of particles from the vacuum intuitively suggests a U-shaped particle worldline, with the two endpoints of the U marking the two particle positions at late times, and with the bottom of the U marking (heuristically) the time of pair creation. We have made this intuition precise, by giving a prescription (3.26) for obtaining the appropriate Feynman propagator iG_F from an integral over particle paths.

4. Power-law FRW

As an example we consider a massless minimally-coupled scalar field in a 3+1-dimensional Robertson-Walker spacetime with a power-law scale factor $a(t) \sim t^c$. We illustrate the formalism of the preceding section by calculating pair production in two ways, using (3.14) and (3.16).

4.1 Mode solution method

The line element reads

$$ds^2 = -dt^2 + a^2(t)d\Sigma_3^2 = \text{const}\,\eta^{\frac{2c}{1-c}}(-d\eta^2 + d\Sigma_3^2)\,, \tag{4.1}$$

where $0 < \eta < \infty$. We take $0 < c < 1$ so the expansion is decelerating. The mode equation (2.4) reads

$$\left(\partial_\eta^2 + k^2 - \frac{\nu^2 - 1/4}{\eta^2}\right) u_k(\eta) = 0\,, \qquad \nu = \frac{3c-1}{2(1-c)}\,. \tag{4.2}$$

We specify the initial state as in section 2.2, by setting

$$\hat{\pi}_k|\text{in}\rangle = i\omega_k\hat{\chi}_k|\text{in}\rangle \qquad \text{at } \eta = \eta_0\,. \tag{4.3}$$

There is a preferred out-vacuum since the field evolves adiabatically at late times. The late-time positive-frequency modes are

$$u_k(\eta) = N\sqrt{\eta}H_\nu^{(2)}(k\eta)\,, \tag{4.4}$$

so that, from (3.14), we find

$$i\bar{\omega} = -\,\partial_\eta \ln \sqrt{\eta}H_\nu^{(2)}(k\eta)\Big|_{\eta=\eta_0}\,. \tag{4.5}$$

This can be inserted into (3.15) to find the particle production parameter,

$$|\gamma|^2 = \left|\frac{\omega - \bar{\omega}^*}{\omega + \bar{\omega}}\right|^2. \tag{4.6}$$

Note that when $\nu^2 = 1/4$ the mode functions reduce to plane waves, and the expression for $|\gamma|$ reduces to

$$|\gamma| = \left|\frac{\omega - k}{\omega + k}\right|. \tag{4.7}$$

4.2 First-quantized approach

In this approach we first need to find the propagator in the reflected potential. This satisfies the differential equation

$$\bar{D}^2(\eta)\bar{G}_F(\eta_1,\eta_2) = -\delta(\eta_1-\eta_2)\,, \tag{4.8}$$

where

$$\bar{D}^2(\eta) = \partial_\eta^2 + k^2 - \begin{cases} \frac{\nu^2-1/4}{\eta^2} & \eta \geq \eta_0 \\ \frac{\nu^2-1/4}{(2\eta_0-\eta)^2} & \eta \leq \eta_0\,. \end{cases} \tag{4.9}$$

The argument of the differential operator can be either η_1 or η_2.

We can solve this equation by holding η_2 fixed and solving the homogeneous equation for η_1 in the two regions which are separated by the delta function. The final Green's function can then be determined by matching across the delta function. This gives

$$\bar{G}_F(\eta_1,\eta_2) = \frac{i\pi}{4c_1}\chi(\eta_<)\left(\sqrt{\eta_>}\,\mathrm{H}_\nu^{(2)}(k\eta_>)\right), \tag{4.10}$$

where $\eta_<(\eta_>)$ is the lesser (greater) of η_1 and η_2. Here $\chi(\eta)$ is defined such that

$$\chi(\eta) = \begin{cases} \sqrt{\eta}\left[c_1\mathrm{H}_\nu^{(1)}(k\eta) + c_2\mathrm{H}_\nu^{(2)}(k\eta)\right] & \eta > \eta_0 \\ \sqrt{2\eta_0-\eta}\,\mathrm{H}_\nu^{(2)}[k(2\eta_0-\eta)] & \eta < \eta_0\,, \end{cases} \tag{4.11}$$

with the coefficients c_1 and c_2 determined by the continuity of $\chi(\eta)$ and its first derivative at η_0.

As in section 3.3, we can easily calculate $\bar{\omega}$ as

$$\begin{aligned} i\bar{\omega} &= \partial_{\eta_2}\ln i\bar{G}_F(\eta_1,\eta_2)\big|_{\eta_1>\eta_2=\eta_0} \\ &= \partial_\eta \ln\chi(\eta)\big|_{\eta_0} \\ &= \partial_\eta \ln\left(\sqrt{2\eta_0-\eta}\,\mathrm{H}_\nu^{(2)}[k(2\eta_0-\eta)]\right)\Big|_{\eta_0} \\ &= -\,\partial_\eta \ln\sqrt{\eta}\,\mathrm{H}_\nu^{(2)}(k\eta)\Big|_{\eta_0}\,. \end{aligned} \tag{4.12}$$

This is precisely what was found in (4.5) from the mode solutions. Moreover the original Green's function can be expressed in terms of (4.10), by using the method of images as in section 3.4.

5. Conclusions

To summarize: given a cosmological background and an initial state characterized by some value of ω, our recipe is to compute the Green's function, $i\bar{G}_F$, in the reflected potential. Particle production is then encoded in

$$i\bar{\omega} = \partial_{\eta_2}\ln i\bar{G}_F(\eta_1,\eta_2)\big|_{\eta_1>\eta_2=\eta_0}\,. \tag{5.1}$$

Since $i\bar{G}_F$ and hence $\bar{\omega}$ can be computed without reference to mode solutions, we have achieved our goal of finding a first-quantized formalism for computing cosmological particle production. The amount of particle production can be directly expressed in terms of ω and $\bar{\omega}$, via

$$\langle N \rangle = \frac{|\gamma|^2}{1-|\gamma|^2} \qquad \text{where} \qquad |\gamma|^2 = \left| \frac{\omega - \bar{\omega}^*}{\omega + \bar{\omega}} \right|^2 . \tag{5.2}$$

Furthermore, the original Green's function can also be cast in terms of an integral over particle worldlines using the method of images.

Finally, let us mention some directions for future work. First and foremost, it would be an interesting challenge to extend our first-quantized formalism to string theory. At a very naive level, the generalization is immediate: merely replace "particle worldlines" with "string worldsheets." In practice, the main difficulty seems to be finding a reflected spacetime which is a legitimate string background. As another potential application, the image charge construction of the propagator could be used to study back-reaction. It would be particularly interesting to study back-reaction in de Sitter space, where the image charge construction of alpha vacua is well known [19, 20].

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